ELSEVIER
Topology and its Applications 115 (2001) 235–238
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

On the local structure of holomorphic foliation singularities ☆

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Received 16 November 1999; received in revised form 17 March 2000

Abstract

Here we prove the following result. Fix integers n, k, s, a_i, $0 \leqslant i \leqslant s$, b_i, $0 \leqslant i \leqslant s$, such that $s \geqslant 0$, $n \geqslant a_0 + 2$, $a_i > a_j \geqslant 0$ for $s \geqslant j > i \geqslant 0$, $a_i \geqslant b_i$ for every i, $b_i > b_{i+1} \geqslant 0$ for $s > i \geqslant 0$. Then there exists a dimension k singular holomorphic foliation F of a neighborhood of $0 \in \boldsymbol{C}^n$ with the following properties. Let Z be the reduction of the singular set of F. Then Z is smooth at 0 and there is a chain of $s+1$ closed smooth submanifolds $0 \in Z_s \subset Z_{s-1} \subset \cdots \subset Z_0 = Z$ such that:

(i) $\dim(Z_i) = a_i$ for every i;

(ii) F has tangential rank b_i at each point of $Z_i \backslash Z_{i+1}$ (with the convention $Z_{-1} := \emptyset$).

AMS classification: 57R30; 58C10; 34A20

Keywords: Holomorphic foliation; Holomorphic foliation with singularities; Singular set of a foliation; Singular set of a holomorphic foliation; Singular set of a singular foliation; Local singular foliation

Let X be a connected complex n-dimensional manifold. A singular dimension k (or codimension $n-k$) holomorphic foliation on X is given assigning a closed analytic subvariety S of X and a dimension k holomorphic foliation of $X \backslash S$, i.e., a rank k holomorphic subbundle E of the tangent bundle $T(X \backslash S)$ closed under Lie bracket: for every $x \in (X \backslash S)$ and every germ $e \in E_x$ we have $[e, e] \in E_x$. Since $T(X \backslash S)$ is the restriction of $T(X)$ to $X \backslash S$, it is a restriction of a vector bundle on X. Thus E extends to a rank k coherent subsheaf, F, of TX [3, Th. 1]. F is uniquely determined if we impose the further condition that $T(X) \backslash F$ has no torsion; given any F the kernel of the quotient map $T(X) \to (T(X)/F)/(\mathit{Tors}(T(X)/F))$ is a saturated extension of E. The corresponding holomorphic foliation with singularities is called saturated and every holomorphic foliation with singularities has a unique saturation. Hence from now on we will always assume to

☆ This research was partially supported by MURST (Italy).
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0166-8641/01/\$ – see front matter
PII: S0166-8641(00)00063-8

have fixed the saturated extension, F, of E. If F is the saturation of the foliation given by E on $X \backslash S$, then F is closed under Lie bracket at every point of S, i.e., for every $x \in X$ and every germ e of F_x we have $[e, e] \in F_x$. For every coherent subsheaf F of $T(X)$, set $S(F) := \{x \in X\colon\ T(X)/F$ is not locally free at $x\}$; $S(F)$ is called the singular set of $T(X)/F$ or of the pair $(F, T(X))$ or (with a bad abuse of notation) of F. The singular set $S(F)$ of the saturation of the holomorphic foliation on $X \backslash S$ is the minimal subset of X at which the associated foliation is not holomorphic. Hence we will call $S(F)$ the singular set of the foliation and we will use freely F to denote the foliation. For any $x \in X$, let $\boldsymbol{m}_x$ be the maximal ideal of the local ring $\boldsymbol{O}_{X,x}$. For any coherent sheaf G on X, let G_x its germ at x and $G(x) := G_x/\boldsymbol{m}_x G_x$ its fiber at x. Thus $G(x)$ is a finite-dimensional $\boldsymbol{C}$-vector space and $\dim(G(x)) = \operatorname{rank}(G)$ if and only if G is locally free in a neighborhood of x (just use Nakayama lemma). For any singular foliation F on X and any $x \in X$ the tangent rank of F at x is the dimension of the image of $F(x)$ into the vector space $T(X)(x)$; if V is a locally closed submanifold of X and $x \in V$ the tangential rank of F along V at x is the dimension of the intersection inside the ambient vector space $T(X)(x)$ of the image of $F(x)$ in $T(X)(x)$ and the $\dim(V)$ vector space $T(V)(x)$. For any F and any locally closed submanifold V of X the function $V \to \mathbb{N}$ sending $x \in V$ into the tangential rank of F along V at x is semicontinuous. The basic reference in this area is the very important paper [1]. For several examples and related results, see [2] and [4, Ch. VI]. In this paper we prove the following result.

Theorem 0.1. *Fix integers n, k, s, a_i, $0 \leqslant i \leqslant s$, b_i, $0 \geqslant i \geqslant s$, such that $s \geqslant 0$, $n \geqslant a_0 + 2$, $a_i > a_j \geqslant 0$ for $s \geqslant j > i \geqslant 0$, $a_i \geqslant b_i$ for every i, $b_i > b_{i+1} \geqslant 0$ for $s > i \geqslant 0$. Then there exists a dimension k singular holomorphic foliation F of a neighborhood of $0 \in \boldsymbol{C}^n$ with the following properties. Let Z be the reduction of the singular set of F. Then Z is smooth at 0 and there is a chain of $s+1$ closed smooth submanifolds $0 \in Z_s \subset Z_{s-1} \subset \cdots \subset Z_0 = Z$ such that*:

(i) $\dim(Z_i) = a_i$ *for every i;*

(ii) *F has tangential rank b_i at each point of $Z_i \backslash Z_{i+1}$ (with the convention $Z_{-1} := \emptyset$).*

Now we discuss the inequalities in the first sentence of the statement of Theorem 0.1. The inequalities $a_i > a_j$ for $i > j$ just means that $Z_i \neq Z_{i+1}$ for every $i < s$ and we may always reduce to this case just decreasing the integer s. The condition $a_i \geqslant b_i$ for every i is obviously a necessary condition by the very definition of tangential rank of a singular foliation along a locally closed submanifold. If $x \in Z_{i+1}$ the tangential rank of F along Z_{i+1} at x is at most the tangential rank of F along Z_i at x which is at most the tangential rank of F along Z_i at a general point of Z_i (semicontinuity); hence we always need to assume $b_i \geqslant b_{i+1}$ for $i < s$; we may assume $b_i > b_{i+1}$ for $i < s$ because we recover the case $b_i = b_{i+1}$ just taking $s' := s - 1$, $Z'_j = Z_j$ for $j \leqslant i$ and $Z'_j = Z_{j+1}$ for $i + 1 \leqslant j \leqslant s - 1$.

To prove Theorem 0.1 we will take a general dimension k holomorphic foliation without singularities on a smooth n-dimensional variety, X_0, obtained from the germ of 0 in $\boldsymbol{C}^n$

by a sequence of at most $2s+2$ blowing-ups of smooth submanifolds. Thus the singular foliation will be very tame: the closure of any leaf is an analytic k-dimensional variety.

1. Proof of Theorem 0.1

Proof of Theorem 0.1. Let Z be the germ at $0 \in \boldsymbol{C}^n$ of an a_s-dimensional smooth manifold. Set $P_s := 0 \in \boldsymbol{C}^n$ all (X_s, Z_s, P_s) the triple $(\boldsymbol{C}^n, Z, 0)$, where X_s and Z_s are seen only as germs near P_s. Let $\pi : Y \to X_s$ be the blowing-up of Z_s and $E := \pi^{-1}(Z_s)$ the exceptional divisor. Take $Q \in \pi^{-1}(P_s)$ and a germ, W, at Q of a section of π over Z_s, i.e., take as W the germ at Q of an a_s-dimensional smooth manifold such that π sends W isomorphically onto the germ Z_s. Let U be a small neighborhood of Q in Y. Consider a general holomorphic submersion $g : U \to \boldsymbol{C}^{n-k}$ with $g(Q) = 0$; in particular we assume that $g^{-1}(0)$ is transversal to W and E. Restricting if necessary the germ U near Q we may assume that g has everywhere differential of rank $n-k$. Thus g induces a smooth dimension k foliation on U. Since Z_s has codimension at least two in X_s, g induces a dimension k singular foliation, G_1, on X_s whose singularity set is contained in Z_s. Since $g^{-1}(0)$ is transversal to W and E, we see that $S(G_1) = Z_s$ near P_s and that $G_1|TZ_s$ has rank $\dim(Z_s) = a_s$ at each point of Z_s near P_s. If $a_s = b_s$ we set $P_{s-1} := Q$, $X_{s-1} := Y$, $F_1 := G_1$, $Z_{s-1} := W$ and call $u : X_{s-1} \to X_{s-1}$ the identity map. Assume $a_s > b_s$. Let B be the germ at Q of a smooth manifold of dimension $n-1-(a_s-b_s)$ such that $W \subset B$, π induces a submersion of B onto Z_0 and W is the image of a section of $\pi|B$. Let $u : X_{s-1} \to Y$ be the blowing-up of Y along B and set $v := \pi \circ u : X_{s-1} \to X_s$. Fix a general $P_{s-1} \in u^{-1}(Q)$ and let Z_{s-1} be a general germ at P_{s-1} of a_s-dimensional submanifold of X_{s-1}. Take a germ at P_{s-1} of a general holomorphic submersion $g_1 : X_1 \to \boldsymbol{C}^{n-k}$ with $g_1(P_{s-1}) = 0$. In particular we assume that $g_1^{-1}(0)$ is transversal to $u^{-1}(B)$, to the strict transform of E and to their intersection. The fibration induced by g_1 is a holomorphic foliation on X_{s-1} without singularities and it induces a singular holomorphic foliation, F_1', on Y and a singular holomorphic foliation, F_1, on X_s. The first part of the proof shows that $\mathit{Sing}(F_1') = B$ and F_1' has tangential rank $a_s - b_s$ along B at each point of B. Thus $Z_s = \mathit{Sing}(F_1)$. For a general submersion g_1 the dimension $(a_s - b_s)$ linear subspace of $TB|\{P_{s-1}\}$ induced by F_1' is transversal to the dimension a_0 linear subspace $TW|\{P_{s-1}\}$. Thus F_1 has tangential rank $a_s - b_s$ at each point of Z_s near P_s. If $s = 0$, we found the local singular foliation claimed by Theorem 0.1. Now assume $s \geqslant 1$. We take the germ at P_s of a smooth a_{s-1}-dimensional submanifold Z_{s-1} of $\boldsymbol{C}^n$ with $Z_s \subset Z_{s-1}$. Call Z_{s-1}' (respectively Z_{-1}'') the strict transform of Z_{s-1} in Y (respectively X_{s-1}). Z_{s-1}' is smooth. Taking Z_{s-1} sufficiently general (or taking sufficiently general B) we obtain the smoothness of Z_{s-1}''. Now we make the previous construction (i.e., two blowing-ups with respect to the integers a_{s-1} and b_{s-1} if $a_{s-1} > b_{s-1}$ and one blowing-up if $a_{s-1} = b_{s-1}$) starting from (X_{s-1}, Z_{s-1}'') instead of (X_s, Z_s). We obtain a singular foliation F_2 on X with $S(F_2) = Z_{s-1}$, tangential rank b_{s-1} at each point of $Z_s \backslash Z_{s-1}$ and such that F_2 has tangential rank b_{s-1} with respect to Z_{s-1} at each point of $Z_{s-1} \backslash Z_s$ (near P).

Claim. *F_2 has tangential rank b_s with respect to Z_s at each point of Z_s.*

Proof. For every $x \in Z_s$ with x near P there is $y \in u^{-1}(x)$ such that u has differential of rank at least $n - b_s$ at y and y is not in the center of one of the two blowing-ups. Thus for the computation of the tangential rank at y of the extension of F_2 with respect of Z_s we may repeat the computation of the tangential rank of F_1 at y with respect to Z_s. □

The claim proves Theorem 0.1 in the case $s = 1$. If $s \geqslant 2$ we may iterate the construction; at each step, say from Z_1 to Z_0 we make two blowing-ups (or just one if $a_0 = b_0$); there is nothing to check on $Z_0 \backslash Z_1$ and the assertions for $Z_i \backslash Z_{i+1}$, $1 \leqslant i < s$, follows as in the proof of the claim. □

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